Semestral

Time:	10:00–13:00, May 11, 2022.	Course name:	*Algebra II*
Degree:	MMath.	Year:	1^{st} Year, 2^{nd} Semester; 2021–2022.
Course instructor:	Ramdin Mawia.	Total Marks:	60.

Attempt four of the following problems, including problems n° 3 & n° 4.

Group Theory

1. Let p be a prime such that $4|p-1$. Describe all groups of order $4p$ (up to isomorphism). **15**
2. Define a solvable group. Show that any group of order p^4q^n is solvable, where $p < q$ are odd primes. **15**
3. Decide whether the following statements are true or false, with brief but complete justifications (**any five**): **15**
   (a) If G is a group of order n and d is a divisor of n, then G has a subgroup of order d.
   (b) A cyclic group may have exactly 14 generators.
   (c) If a finite group G acts freely and transitively on a finite set X, then G and X have the same cardinality.
   (d) The number of Sylow p-subgroups of $\mathrm{GL}_2(\mathbb{F}_p)$ is $p+1$.
   (e) A finite group can be equal to the union of conjugates of a proper subgroup.
   (f) If $1 \to F \to E \to G \to 1$ is a short exact sequence of groups with F and G cyclic, then E is necessarily abelian.
   (g) If P and Q are Sylow p-subgroups of G with $|P| = |Q| = p^3$ and $|P \cap Q| \geqslant p^2$, then $P = Q$.

Galois Theory

4. When do we say that a field extension is separable? **15**
   (a) Define the separable degree and prove that it divides the degree of the extension (for finite extensions).
   (b) Is it true that every finite extension of a finite field is separable? Justify.
5. Give two equivalent definitions of a Galois extension (without proving that they are indeed equivalent). State and prove the Fundamental Theorem of Galois Theory. **15**
6. Give at least four equivalent definitions of a separable polynomial and prove their equivalence. What do we mean by the Galois group of such a polynomial? Find the Galois groups of two of the following polynomials, with justifications: **15**
   (a) $X^4 + 5X + 10 \in \mathbb{Q}[X]$.
   (b) $X(X^2+20)(X^2-4)+2 \in \mathbb{Q}[X]$.
   (c) $X^3 + 3X + 2 \in \mathbb{Q}[X]$.
7. State whether the following statements are true or false, with brief but complete justifications (**any five**): **15**
   (a) Any finite extension of a finite field is cyclic (i.e., Galois with cyclic Galois group).
   (b) If $K = k[\alpha]$ is a simple n-radical extension (i.e., $\alpha^n \in k$) and if K contains a primitive n^{th} root of unity, then K/k is a cyclic extension.
   (c) The number $\cos(\pi/7)$ is constructible.
   (d) Roots of the polynomial $X(X^2+2022)(X^2-16)+2 \in \mathbb{Q}[X]$ are expressible in radicals.
   (e) No finite field is algebraically closed.
   (f) The algebraic closure of $\mathbb{Q}$ and that of $\mathbb{Q}[\sqrt{5}]$ are isomorphic.
   (g) -2 is a square in the field $\mathbb{Q}[\sqrt[4]{-3}]$.